Screening, and Landau → Lenard-Balescu: (toward)

- seek remove ad-hoc treatment of infrared divergence via physically motivated cut-off

⇒ Seek screened Landau result

So by:

① Calculate $\hat{\phi}$ due screened test particle at velocity v'

then

② calculate deflection of particle of velocity $\underline{v}$ due $\hat{\phi}$

For ①, $$-\underline{\nabla}\cdot\underline{\underline{\epsilon}}\cdot\underline{\nabla}\phi = 4\pi e\,\delta(\underline{x}-\underline{v}'t)$$

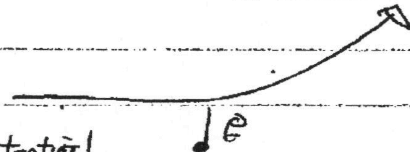
$$\hat{\phi}_{\underline{k},\omega} = \frac{4\pi e}{k^2\,\epsilon(k,\omega)}\,2\pi\,\delta(\omega-\underline{k}\cdot\underline{v}')$$

$$\phi_{\underline{k}}(t) = \int \frac{d\omega}{(2\pi)} \frac{4\pi e'}{k^2 \epsilon(k,\omega)} 2\pi\, \delta(\omega - \underline{k}\cdot \underline{v}')\, e^{-i\omega t}$$

$$= \frac{4\pi e'}{k^2 \epsilon(k, \underline{k}\cdot\underline{v}')} e^{-i\underline{k}\cdot\underline{v}' t}$$

potential.

For deflection:



$$\underline{q} = \int_{u.p.o.} dt \left(- \frac{\overbrace{\partial U}^{\text{interaction potential}}}{\partial \underline{r}} \right) = - \int_{\underline{r} = \underline{\rho} + \underline{v}t} \frac{\partial U}{\partial r}$$

$\underline{\rho}$ → impact param.

$$U = e\phi'$$

follows previous

$$= 4\pi e e' \int_{upo} \frac{d^3k\, e^{i\underline{k}\cdot\underline{r}}\, e^{-i\underline{k}\cdot\underline{v}'t}}{k^2 \epsilon(k, \underline{k}\cdot\underline{v}')}$$

$$= 4\pi e e' \int \frac{d^3k\, e^{i\underline{k}\cdot\underline{\rho}}\, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}}{k^2 [\epsilon(k, \underline{k}\cdot\underline{v}')]}$$

$\Rightarrow$

$$\underline{q} = 4\pi e e' \int \frac{d^3k}{(2\pi)^3} \frac{-i\underline{k}\, e^{i\underline{k}\cdot\underline{\rho}}}{k^2 \epsilon(k, \underline{k}\cdot\underline{v}')} 2\pi\, \delta(\underline{k}\cdot(\underline{v} - \underline{v}'))$$

from:

$$\int dt\, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}$$

using $\delta(\underline{k}\cdot(\underline{v}-\underline{v}')) = \delta(k_\parallel |\underline{v}-\underline{v}'|)$

$$= \frac{1}{|\underline{v}-\underline{v}'|}\,\delta(k_\parallel)$$

$\therefore$

$$\boxed{\underline{q} = 4\pi e e' \int \frac{d^2k_\perp}{(2\pi)^2}\, \frac{-i\underline{k}_\perp\, e^{i\underline{k}_\perp\cdot\underline{\rho}}}{k^2\, \epsilon(\underline{k}, \underline{k}\cdot\underline{v}')\, |\underline{v}-\underline{v}'|}}$$

momentum transfer

coulomb — screen — passage time

for $B_{\alpha\beta}$

$$B_{\alpha\beta} = \int d\Gamma_1\, \frac{q_\alpha q_\beta}{2}\, |\underline{v}-\underline{v}'|$$

$$= \int d^2\rho\; q_\alpha q_\beta\, |\underline{v}-\underline{v}'|$$

$$\int d^2\rho\, q_\alpha q_\beta \sim \int d^2\rho\; e^{i\underline{k}_\perp\cdot\underline{\rho}}\, e^{i\underline{k}_\perp'\cdot\underline{\rho}}$$

$$\sim (2\pi)^2\, \delta(\underline{k}_\perp + \underline{k}_\perp')$$

$$\int d^2k_\perp'\, \delta(\underline{k}_\perp + \underline{k}_\perp') = 1$$

so

$$B_{\alpha,\beta} = 2e^2 e'^2 \int d^2 k_\perp \frac{k_{\perp\alpha} \, k_{\perp\beta}}{|k_\perp^2 \, \epsilon(\underline{k}_\perp, \underline{k}_\perp \cdot \underline{v}')|^2 \, |\underline{v}-\underline{v}'|}$$

$\Rightarrow$

$$\boxed{B_{\alpha,\beta} = 2e^2 e'^2 \int d^2 k_\perp \frac{k_{\perp\alpha} \, k_{\perp\beta}}{|k_\perp^2 \, \epsilon(\underline{k}_\perp, \underline{k}_\perp \cdot \underline{v}')|^2 \, |\underline{v}-\underline{v}'|}}$$

Note:

i.) $\epsilon(\underline{k}, \underline{k} \cdot \underline{v}')$ $\rightarrow$ dynamic screening factor
$\rightarrow$ evaluates ϵ induced by (ballistically) propagating source

ii.) note if $\epsilon \rightarrow 1$ (no collective screening)

$$B \sim \int d^2 k \frac{k_\perp^2}{|\epsilon|^2 k_\perp^4} \sim \int dk_\perp \, k_\perp \frac{k_\perp^2}{k_\perp^4} \sim \int \frac{dk_\perp}{k_\perp}$$

$$\sim \ln(k_{\perp max} / k_{\perp min})$$

$\rightarrow$ recovers Coulomb logarithm.

if $\underline{k}, \omega \rightarrow 0$

$$\epsilon = 1 + 1/k^2 \lambda_D^2$$

$k_\perp^2 \epsilon \sim k_\perp^2 + 1/\lambda_D^2$ $\rightarrow$ no long range divergence iii.) limits of integration:

$k_{min} \sim 1/\lambda_D$ (via ϵ)

$k_{max} \sim \frac{1}{2}\mu V_{rel}^2/ee'$ (distance of closest approach)

Now, can re-write $B_{\alpha,\beta}$ as

$$B_{\alpha,\beta} = 2(ee')^2 \int_{-\infty}^{+\infty} d\omega \int_{k<k_{max}} d^3k \; \delta(\omega - \underline{k}\cdot\underline{V})\,\delta(\omega - \underline{k}\cdot\underline{V}') \frac{k_\alpha k_\beta}{k^4 |\epsilon(k,\omega)|^2}$$

recovers L-B. theory noting:

one $\delta(\omega - k\cdot V')$ $\Rightarrow$ propagator in $\langle \tilde{f}\tilde{f}\rangle_{k,\omega}$

2^{nd} $\delta(\omega - \underline{k}\cdot\underline{V})$ $\Rightarrow$ propagator in Q.L. terms.

→ properties of Landau Collision Integral

n.b.: Read Kulsrud 8.1 → 8.3 ⟷ different approach

now, switching from $p \to \underline{v}$

$$\left.\frac{\partial \langle f \rangle}{\partial t}\right|_c = -\frac{\partial}{\partial \underline{v}} \cdot \underline{J}(\underline{v})$$

$$\underline{J}(\underline{v}) = \sum_{spc} \int d^3v' \, B_{\alpha\beta} \left[\frac{\partial f(\underline{v}')}{\partial v'_\beta} f(\underline{v}) - f(\underline{v}') \frac{\partial f(\underline{v})}{\partial v_\beta} \right]$$

$$B_{\alpha\beta} = \frac{2\pi (q q')^2}{\mu_p^2 |v - v'|} \ln \Lambda \left[\delta_{\alpha\beta} - \frac{v_{\alpha rel} v_{\beta rel}}{v_{rel}^2} \right]$$

$$v - v' = v_{rel}$$

then for electrons,

$$\left.\frac{\partial \langle f \rangle}{\partial t}\right|_c = -\frac{\partial}{\partial v} J(\underline{v})$$

$$= -\frac{\partial}{\partial \underline{v}} \cdot \Big[\underbrace{J_{e,e}(\underline{v})}_{\text{electrons as field particles}} + \underbrace{J_{e,i}(\underline{v})}_{\text{ions as field particles}} \Big]$$

$$\underline{J}(v)_\alpha = \int d^3v_e' \, B_{\alpha\beta \atop e,e} \left[\frac{\partial f_e(\underline{v}_e')}{\partial v_{e\beta}'} f_e(v) - f_e(v_e') \frac{\partial f_e(v)}{\partial v_\beta} \right]$$

$$+ \int d^3v_i' \, B_{\alpha\beta \atop e,i} \left[\frac{\partial f_i(v_i')}{\partial v_{i\beta}'} f_e(\underline{V}) - f_i(v_i') \frac{\partial f_e(\underline{v})}{\partial v_\beta} \right]$$

$$B_{\alpha\beta \atop e,e} = \frac{2\pi e^2}{\mu_{e,e} |v_{rel}|} \ln \Lambda_{e,e} \left[\delta_{\alpha\beta} - \frac{v_{rel\,\alpha} v_{rel\,\beta}}{v_{rel}^2} \right]$$

$$B_{\alpha\beta \atop e,i} = \frac{2\pi e^2}{\mu_{e,i} |v_{rel}|} \ln \Lambda_{e,i} \left[\delta_{\alpha\beta} - \frac{v_{rel\,\alpha} v_{rel\,\beta}}{v_{rel}^2} \right]$$

→ negligible difference in magnitude

Note can simplify form to:

$$\left.\frac{\partial f}{\partial t}\right)_c = \underbrace{\frac{2\pi e^4 \ln\Lambda}{m_e^2}}_{\text{lead factor}} \frac{\partial}{\partial \underline{v}} \cdot \underset{\uparrow \atop \text{current (sign flipped)}}{\underline{J}(\underline{v})}$$

$$\underline{J}(\underline{v}) = \int d^3v' \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\underset{\uparrow \atop \text{diff'n}}{\frac{\partial f(\underline{v})}{\partial \underline{v}}} f(\underline{v}') - \underset{\uparrow \atop \text{drag}}{\frac{\partial f(\underline{v}')}{\partial \underline{v}'}} f(\underline{v}) \right]$$

Now, need demonstrate several properties:

(a)
- H theorem

(b)
- conservation for like species (a/a' L-B.)

Now, H theorem:

→ entropy increases, except for Maxwellian

→ $\frac{dH}{dt} \leq 0$ $\quad H = -\int f \ln f \, d^3v$
$\quad = +S$

need show $H \downarrow$ for $S \uparrow$

$$\frac{dH}{dt} = \int d^3v \left[C(f) \ln f + C(f) \right]$$

$$C(f) = -\nabla_v \cdot J$$

$$\frac{dH}{dt} = \int d^3v \left[C(f) \ln f + C(f) \right]$$

$$= \int d^3v \left[-\nabla_v \cdot \underline{J}(\underline{v}) \ln f + \cancel{\nabla_v \cdot \underline{J}(\underline{v})} \right]$$

conservation

$$= +\int d^3v \, \frac{1}{f} \frac{\partial f}{\partial \underline{v}} \cdot \underline{J}(\underline{v})$$

so

$$\frac{dH}{dt} = (\#)\int d^3v \int d^3v' \, \frac{1}{f} \frac{\partial f}{\partial \underline{v}} \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right]$$

now $\underline{v} \leftrightarrow \underline{v}'$ and add $\Rightarrow \underline{J}$ is odd in interchange

so

$$\frac{dH}{dt} = (\#)\int d^3v \int d^3v' \left(\frac{1}{f(\underline{v})} \frac{\partial f}{\partial \underline{v}} - \frac{1}{f(\underline{v}')} \frac{\partial f}{\partial \underline{v}'} \right) \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right] (\cancel{+}\tfrac{1}{2})$$

$$= \# \int d^3v \int d^3v' \, \frac{1}{f(\underline{v}) f(\underline{v}')} \left(f(\underline{v}') \frac{\partial f}{\partial \underline{v}} - f(\underline{v}) \frac{\partial f}{\partial \underline{v}'} \right) \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right] (\cancel{+}\tfrac{1}{2})$$

$$= -\int d^3v' \int d^3v \, \frac{1}{f(\underline{v}) f(\underline{v}')} \left(\quad \right) \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left(\quad \right)$$

where $\langle\ \rangle = \left[f(\underline{v}') \dfrac{\partial f}{\partial \underline{v}} - f(\underline{v}) \dfrac{\partial f}{\partial \underline{v}'} \right]$

so $\dfrac{dH}{dt} \geq 0$ → entropy increases

Note → $\dfrac{dH}{dt} = 0$ for Maxwellian

→ if e,i interaction, note $dH/dt = 0$ if both electrons, ions Maxwellian.